On Potentially 3-regular graph graphic Sequences *

Lili Hu , Chunhui Lai
Department of Mathematics, Zhangzhou Teachers College,
Zhangzhou, Fujian 363000, P. R. of CHINA.
jackey2591924@163.com (Lili Hu)
zjlaichu@public.zzptt.fj.cn(Chunhui Lai, Corresponding author)

Abstract

For given a graph H, a graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be potentially H-graphic if there exists a realization of π containing H as a subgraph. In this paper, we characterize the potentially H-graphic sequences where H denotes 3-regular graph with 6 vertices. In other words, we characterize the potentially $K_{3,3}$ and $K_6 - C_6$-graphic sequences where $K_{r,r}$ is an $r \times r$ complete bipartite graph. One of these characterizations implies a theorem due to Yin [25].
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1 Introduction

We consider finite simple graphs. Any undefined notation follows that of Bondy and Murty [1]. The set of all non-increasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is denoted by NS_n. A sequence $\pi \epsilon NS_n$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred as a realization of π. The set of all graphic sequence in NS_n is denoted by GS_n. A graphic sequence π is potentially H-graphic if there is a realization of π containing H as a subgraph. Let C_k

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and P_k denote a cycle on k vertices and a path on $k+1$ vertices, respectively. Let $\sigma(\pi)$ the sum of all the terms of π, and let $[x]$ be the largest integer less than or equal to x. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph. Let $G-H$ denote the graph obtained from G by removing the edges set $E(H)$ where H is a subgraph of G. In the degree sequence, r^t means r repeats t times, that is, in the realization of the sequence there are t vertices of degree r.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted by $ex(n,H)$, and is known as the Turán number. In terms of graphic sequences, the number $2ex(n,H)+2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi)\geq l$ is forcibly H-graphical. Gould, Jacobson and Lehel [3] considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H,n)$ such that every n-term positive graphic sequence $\pi=(d_1,d_2,\cdots,d_n)$ with $\sigma(\pi)\geq\sigma(H,n)$ has a realization G containing H as a subgraph. They proved that $\sigma(pK_2,n)=(p-1)(2n-p)+2$ for $p\geq 2$; $\sigma(C_4,n)=2[\frac{3n-1}{2}]$ for $n\geq 4$. Erdös, Jacobson and Lehel [2] showed that $\sigma(K_k,n)\geq(k-2)(2n-k+1)+2$ and conjectured that the equality holds. In the same paper, they proved the conjecture is true for $k=3$ and $n\geq 6$. The conjecture is confirmed in [3] and [15]-[18]. Ferrara, Gould and Schmitt proved the conjecture [4] and they also determined in [5] $\sigma(F_k,n)$ where F_k denotes the graph of k triangles intersecting at exactly one common vertex. Recently, Li and Yin [20] further determined $\sigma(K_r,n)$ for $r\geq 7$ and $n\geq 2r+1$. The problem of determining $\sigma(K_r,n)$ is completely solved. [24-27] determined $\sigma(K_{r,s},n)$ for $s\geq r\geq 1$ and sufficiently large n. Yin, Li, and Mao [29] determined $\sigma(K_{r+1}-e,n)$ for $r\geq 3$ and $r+1\leq n\leq 2r$ and $\sigma(K_5-e,n)$ for $n\geq 5$. Yin and Li[28] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1}-e,n)$ for $r\geq 2$ and $n\geq 3r^2-r-1$ (In fact, Yin and Li[28] also determining the values $\sigma(K_{r+1}-ke,n)$ for $r\geq 2$ and $n\geq 3r^2-r-1$). After reading[28], using Yin-Li method Yin [32] determined $\sigma(K_{r+1}-K_3,n)$ for $n\geq 3r+5, r\geq 3$. Yin, Chen and Schmitt [31] determined $\sigma(F_{t,r,k},n)$ for $k\geq 2$, $t\geq 3$, $1\leq r\leq t-2$ and n sufficiently large. Lai [10-12] determined $\sigma(K_5-C_4,n)$, $\sigma(K_5-P_3,n)$, $\sigma(K_5-P_4,n)$ and $\sigma(K_5-K_3,n)$ for $n\geq 5$. Determining $\sigma(K_{r+1}-H,n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4\not\subset C_i$, but $P_3\subset C_i$ for $i\geq 5$). So, after reading[28] and

[32], using Yin-Li method Lai and Hu [13] determined $\sigma(K_{r+1}-H,n)$ for $n\geq 4r+10$, $r\geq 3$, $r+1\geq k\geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not contain a cycle on 3 vertices and $\sigma(K_{r+1}-P_2,n)$ for $n\geq 4r+8$, $r\geq 3$. Using Yin-Li method Lai [14] determined $\sigma(K_{r+1}-Z_4,n)$, $\sigma(K_{r+1}-(K_4-e),n)$, $\sigma(K_{r+1}-K_4,n)$ for $n\geq 5r+16$, $r\geq 4$ and $\sigma(K_{r+1}-Z,n)$ for $n\geq 5r+19$, $r+1\geq k\geq 5$, $j\geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contain a cycle on 4 vertices.

A harder question is to characterize the potentially H-graphic sequences without zero terms. Luo [21] characterized the potentially C_k-graphic sequences for each $k=3,4,5$. Recently, Luo and Warner [22] characterized the potentially K_4-graphic sequences. Eschen and Niu [23] characterized the potentially K_4-e-graphic sequences. Yin and Chen [30] characterized the potentially $K_{r,s}$-graphic sequences for $r=2,s=3$ and $r=2,s=4$. Yin et al. [33] characterized the potentially K_5-e, K_6-e and K_6-graphic sequences. Hu and Lai [6-8] characterized the potentially K_5-C_4, K_5-P_4 and K_5-E_3-graphic sequences where E_3 denotes graphs with 5 vertices and 3 edges. In this paper, we characterize the potentially H-graphic sequences where H denotes 3-regular graph with 6 vertices. In other words, we characterize the potentially $K_{3,3}$ and K_6-C_6-graphic sequences where $K_{r,r}$ is an $r\times r$ complete bipartite graph. One of these characterizations implies a theorem due to Yin [25].

2 Preparations

Let $\pi=(d_1,\cdots,d_n)\epsilon NS_n, 1\leq k\leq n$. Let

$$\pi_k''=\begin{cases}(d_1-1,\cdots,d_{k-1}-1,d_{k+1}-1,\cdots,d_{d_k+1}-1,d_{d_k+2},\cdots,d_n),\\ \text{ if } d_k\geq k,\\ (d_1-1,\cdots,d_{d_k}-1,d_{d_k+1},\cdots,d_{k-1},d_{k+1},\cdots,d_n),\\ \text{if } d_k<k.\end{cases}$$

Denote $\pi_k'=(d_1',d_2',\cdots,d_{n-1}')$, where $d_1'\geq d_2'\geq\cdots\geq d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k''. Then π_k' is called the residual sequence obtained by laying off d_k from π. For simplicity, we denote π_n' by π' in this paper.

For a nonincreasing positive integer sequence $\pi=(d_1,d_2,\cdots,d_n)$, we write $m(\pi)$ and $h(\pi)$ to denote the largest positive terms of π and the smallest positive terms of π, respectively. We need the following results.

Theorem 2.1 [**3**] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

Theorem 2.2 [**19**] If $\pi = (d_1, d_2, \cdots, d_n)$ is a sequence of nonnegative integers with $1 \leq m(\pi) \leq 2$, $h(\pi) = 1$ and even $\sigma(\pi)$, then π is graphic.

Theorem 2.3 [**30**] Let $n \geq 5$ and $\pi = (d_1, d_2, \cdots, d_n) \epsilon GS_n$. Then π is potentially $K_{2,3}$-graphic if and only if π satisfies the following conditions:

(1) $d_2 \geq 3$ and $d_5 \geq 2$;

(2) If $d_1 = n-1$ and $d_2 = 3$, then $d_5 = 3$;

(3) $\pi \neq (3^2, 2^4)$, $(3^2, 2^5)$, $(4^3, 2^3)$, $(n-1, 3^5, 1^{n-6})$ and $(n-1, 3^6, 1^{n-7})$.

Theorem 2.4 [**7**] Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $K_5 - P_4$-graphic if and only if the following conditions hold:

(1) $d_2 \geq 3$ and $d_5 \geq 2$.

(2) $\pi \neq (n-1, k, 2^t, 1^{n-2-t})$ where $n \geq 5$, $k, t = 3, 4, \cdots, n-2$, and, k and t have different parities.

(3) For $n \geq 5$, $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \cdots, n-2k$ and $k = 1, 2, \cdots, [\frac{n-1}{2}] - 1$.

(4) $\pi \neq (3^2, 2^4)$ and $(3^2, 2^5)$.

Lemma 2.5 (Kleitman and Wang [9]) π is graphic if and only if π' is graphic.

The following corollary is obvious.

Corollary 2.6 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

3 Main Theorems

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 6$. Then π is potentially $K_{3,3}$-graphic if and only if the following conditions hold:

(1) $d_6 \geq 3$;

(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 4$;

(3) $d_2 = n-1$ implies $d_3 \geq 5$ or $d_6 \geq 4$;

(4) $d_1 + d_2 = 2n - i$ and $d_{n-i+3} = 1 (3 \leq i \leq n-4)$ implies $d_3 \geq 5$ or $d_6 \geq 4$;

(5) $d_1 + d_2 = 2n - i$ and $d_{n-i+4} = 1 (4 \leq i \leq n-3)$ implies $d_3 \geq 4$;

(6) $\pi = (d_1, d_2, 3^4, 2^t, 1^{n-6-t})$ or $(d_1, d_2, 4^2, 3^2, 2^t, 1^{n-6-t})$ implies $d_1 + d_2 \leq n + t + 2$;

(7) $\pi = (d_1, d_2, 4, 3^4, 2^t, 1^{n-7-t})$ implies $d_1 + d_2 \leq n + t + 3$;

(8) For $t = 5, 6$, $\pi \neq (n-i, k+i, 4^t, 2^{k-t}, 1^{n-2-k})$ where $i = 1, \cdots, [\frac{n-k}{2}]$ and $k = t, \cdots, n-2i$;

(9) $\pi \neq (5^4, 3^2, 2)$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$, $(n-3, 3^7, 1^{n-8})$.

Proof: First we show the conditions (1)-(9) are necessary conditions for π to be potentially $K_{3,3}$-graphic. Assume that π is potentially $K_{3,3}$-graphic. (1) is obvious. Let G be a realization of π which contains $K_{3,3}$ and let $v_i \epsilon V(G)$ with degree $d(v_i) = d_i$ for $i = 1, 2$. Then $G - v_1$ contains $K_{2,3}$. Thus, $G - v_1$ contains at least two vertices with degree at least 3. Therefore, $d_1 = n - i, i = 1, 2$ implies $d_{4-i} \geq 4$. Hence, (2) holds. Clearly, $G - v_1 - v_2$ contains $K_{1,3}$ or $K_{2,2}$. If $G - v_1 - v_2$ contains $K_{1,3}$ and $d_2 = n - 1$, then $d_3 \geq 5$. If $G - v_1 - v_2$ contains $K_{2,2}$ and $d_2 = n - 1$, then $d_6 \geq 4$. Hence, (3) holds. Now suppose $G - v_1 - v_2$ contains $K_{1,3}$ and denote the vertex with degree 3 in $K_{1,3}$ by v_3. If $d_1 + d_2 = 2n - i$ and $d_{n-i+3} = 1$, then we will show that both v_1 and v_2 are adjacent to v_3, i.e., $d_3 \geq 5$. By way of contradiction, if v_1 or v_2 is not adjacent to v_3, then $2n-i = d_1+d_2 \leq 9+2(n-i-4)+i-2$, i.e., $0 \leq -1$, a contradiction. Hence, both v_1 and v_2 are adjacent to v_3, i.e., $d_3 \geq 5$. Similarly, if $G - v_1 - v_2$ contains $K_{2,2}$ and $d_1 + d_2 = 2n - i$, $d_{n-i+3} = 1$, then $d_6 \geq 4$. Hence, (4) holds. With the same argument as above, one can show that (5) holds. If $\pi = (d_1, d_2, 3^4, 2^t, 1^{n-6-t})$ is potentially $K_{3,3}$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_{3,3}$ as a subgraph so that the vertices of $K_{3,3}$ have the largest degrees of π. Therefore, the sequence $\pi_1 = (d_1 - 3, d_2 - 3, 2^t, 1^{n-6-t})$ obtained from $G - K_{3,3}$ is graphic. It follows $d_1 - 3 + d_2 - 3 \leq 2t + n - 6 - t + 2$, i.e., $d_1 + d_2 \leq n + t + 2$. Similarly, one can show that $\pi = (d_1, d_2, 4^2, 3^2, 2^t, 1^{n-6-t})$ also implies $d_1 + d_2 \leq n+t+2$ and $\pi = (d_1, d_2, 4, 3^4, 2^t, 1^{n-7-t})$ implies $d_1 + d_2 \leq n+t+3$. Hence, π satisfies (6) and (7). If $\pi = (n - i, k + i, 4^5, 2^{k-5}, 1^{n-2-k})$ is potentially $K_{3,3}$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_{3,3}$ as a subgraph so that the vertices of $K_{3,3}$ have the largest degrees of π. Therefore, the sequence $\pi_2 = (n - i - 3, k + i - 3, 1^4, 4, 2^{k-5}, 1^{n-2-k})$ obtained from $G - K_{3,3}$ must be graphic. It

follows $n-i-3+k+i-3+4+4-12 \leq 2(k-5)+n-2-k$, i.e., $-10 \leq -12$, a contradiction. Hence, $\pi \neq (n-i, k+i, 4^5, 2^{k-5}, 1^{n-2-k})$. Similarly, one can show that $\pi \neq (n-i, k+i, 4^6, 2^{k-6}, 1^{n-2-k})$. Hence, (8) holds. Now it is easy to check that $(5^4, 3^2, 2)$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$ and $(3^8, 1^2)$ are not potentially $K_{3,3}$-graphic. Since $(3^2, 2^4)$, $(3^2, 2^5)$ and $(4^3, 2^3)$ are not potentially $K_{2,3}$-graphic by Theorem 2.3, we have $\pi \neq (n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$ and $(n-1, 5^3, 3^3, 1^{n-7})$. If $\pi = (n-2, 4, 3^5, 1^{n-7})$ is potentially $K_{3,3}$-graphic, then according to Theorem 2.1, there exists a realization G of π containing $K_{3,3}$ as a subgraph so that the vertices of $K_{3,3}$ have the largest degrees of π. Therefore, the sequence $\pi^* = (n-5, 3, 1^{n-6})$ obtained from $G - K_{3,3}$ must be graphic. It follows the sequence $\pi_1^* = (2)$ should be graphic, a contradiction. Hence, $\pi \neq (n-2, 4, 3^5, 1^{n-7})$. Similarly, one can show that $\pi \neq (n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$ and $(n-3, 3^7, 1^{n-8})$. Hence, (9) holds.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1)-(9). Our proof is by induction on n. We first prove the base case where $n = 6$. Since $\pi \neq (4^6)$, then π is one of the following: (5^6), $(5^4, 4^2)$, $(5^3, 4^2, 3)$, $(5^3, 3^3)$, $(5^2, 4^4)$, $(5, 4^4, 3)$, $(5, 4^2, 3^3)$, $(4^4, 3^2)$, $(4^2, 3^4)$, (3^6). It is easy to check that all of these are potentially $K_{3,3}$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 7)$, we will show that π is potentially $K_{3,3}$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. It is easy to check that π' satisfies (1), (2) and (7). If π' also satisfies (3), (6) and (8)-(9), then by the induction hypothesis, π' is potentially $K_{3,3}$-graphic, and hence so is π.

If π' does not satisfy (3), i.e., $d_2' = n-2$, $d_3' = 4$ and $d_6' = 3$. Then $d_1 = d_2 = n-1$, $d_3 = 4$ and $7 \leq n \leq 8$. Hence, $\pi = (6^2, 4^5)$ or $(7^2, 4^6)$, which is impossible by (8).

If π' does not satisfy (6), then π' is just $(5^2, 4^2, 3^2)$, and hence $\pi = (6^2, 4^5)$, which is impossible by (8).

If π' does not satisfy (8), then π' is just $(6^2, 4^5)$ or $(7^2, 4^6)$, and hence $\pi = (7^2, 5^2, 4^4)$ or $(8^2, 5^2, 4^5)$. Since $\pi_1' = (6, 4^2, 3^4)$ or $(7, 4^2, 3^5)$ is potentially $K_{2,3}$-graphic, π is potentially $K_{3,3}$-graphic.

If π' does not satisfy (9), then π' is just (4^6), and hence $\pi = (5^4, 4^3)$. It is easy to see that π is potentially $K_{3,3}$-graphic.

Case 2: $d_n = 3$. Consider $\pi' = (d_1', d_2', \cdots, d_{n-1}')$ where $d_{n-4}' \geq 3$ and $d_{n-1}' \geq 2$. If π' satisfies (1)-(3) and (6)-(9), then by the induction

hypothesis, π' is potentially $K_{3,3}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_6 = 2$, then $d_3 = \cdots = d_n = 3$. Since $d'_{n-4} \geq 3$, we have $7 \leq n \leq 9$. If $n = 7$, then $\pi = (d_1, d_2, 3^5)$ where $3 \leq d_2 \leq d_1 \leq 6$. Since $\sigma(\pi)$ is even, $\pi = (4, 3^6)$, $(6, 3^6)$, $(5, 4, 3^5)$ or $(6, 5, 3^5)$, which is impossible by (2) and (9). If $n = 8$, then $\pi = (d_1, 3^7)$ where $3 \leq d_1 \leq 7$ and d_1 is odd. Hence, $\pi = (3^8)$, $(5, 3^7)$ or $(7, 3^7)$, which is also impossible by (2) and (9). If $n = 9$, then $\pi = (3^9)$, a contradiction.

If π' does not satisfy (2), i.e., $d'_1 = n - 1 - i$ and $d'_{4-i} = 3$ for $i = 1, 2$. If $d'_1 = n - 2$ and $d'_3 = 3$, then $d_1 = n - 1$ and $d_3 = 4$. Since $\sigma(\pi)$ is even, we have $d_4 = 3$. Hence, $\pi = (n - 1, d_2, 4, 3^{n-3})$ where $4 \leq d_2 \leq n - 2$ and d_2 is even. If $d_2 = 4$, then $\pi = (n - 1, 4^2, 3^{n-3})$. By $\pi \neq (6, 4^2, 3^4)$ and $(7, 4^2, 3^5)$, we have $n \geq 9$. Since $\pi'_1 = (3^2, 2^{n-3})$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic. If $5 \leq d_2 \leq n - 2$, then $\pi'_1 = (d_2 - 1, 3, 2^{n-3})$ is also potentially $K_{2,3}$-graphic by Theorem 2.3. Hence, π is potentially $K_{3,3}$-graphic. If $d'_1 = n - 3$ and $d'_2 = 3$, then $d_1 = n - 2$, $d_2 = 4$ and $3 \leq d_3 \leq 4$. Since $\sigma(\pi)$ is even, $d_3 = 3$. Hence, $\pi = (n - 2, 4, 3^{n-2})$ where n is arbitrary. Since $\pi \neq (5, 4, 3^5)$ and $(6, 4, 3^6)$, we have $n \geq 9$. We will show that π is potentially $K_{3,3}$-graphic. It is enough to show $\pi_1 = (n - 5, 3^{n-6}, 1)$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-6})$ is graphic. Clearly, C_{n-6} is a realization of π_2.

If π' does not satisfy (3), i.e., $d'_2 = n - 2$, $d'_3 = 4$ and $d'_6 = 3$. It is easy to check that $d_1 = d_2 = n - 1$ and $4 \leq d_3 \leq 5$. If $d_3 = 4$, then by (3), we have $\pi = ((n-1)^2, 4^4, 3^{n-6})$ where n is even. Since $\pi'_1 = (n-2, 3^4, 2^{n-6})$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic. If $d_3 = 5$, then $\pi = ((n-1)^2, 5, 4^k, 3^{n-3-k})$ where $0 \leq k \leq 2$, n and k have the same parity. Since $\pi'_1 = (n - 2, 4, 3^k, 2^{n-3-k})$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic.

If π' does not satisfy (6), then π' is just $(5^2, 3^4)$, $(6^2, 3^4, 2)$ or $(5^2, 4^2, 3^2)$. Since $\pi \neq (6^2, 4, 3^4)$, $(7^2, 3^6)$, $(6^2, 4^3, 3^2)$ and $(6, 5^3, 3^3)$, then $\pi = (6^2, 5, 4, 3^3)$ which is potentially $K_{3,3}$-graphic.

If π' does not satisfy (7), then π' is just $(6^2, 4, 3^4)$ and hence $\pi = (7^2, 5, 3^5)$ or $(7^2, 4^2, 3^4)$. But $\pi = (7^2, 4^2, 3^4)$ contradicts condition (3), thus $\pi = (7^2, 5, 3^5)$. Since $\pi'_1 = (6, 4, 2^5)$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic.

If π' does not satisfy (8), then π' is just $(6^2, 4^5)$ or $(7^2, 4^6)$, and hence $\pi = (7^2, 5, 4^4, 3)$ or $(8^2, 5, 4^5, 3)$. Since $\pi'_1 = (6, 4, 3^4, 2)$ or $(7, 4, 3^5, 2)$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic.

If π' does not satisfy (9), since $\pi \neq (4^2, 3^6)$ and $(4, 3^8)$, then π' is one of the following: (4^6), $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, (3^8), $(4, 3^8)$, $(3^8, 2)$, $(6, 4^2, 3^4)$, $(7, 4^2, 3^5)$, $(6, 5^3, 3^3)$, $(5, 4, 3^5)$, $(6, 4, 3^6)$, $(4, 3^6)$, $(5, 3^7)$. Since $\pi \neq (6^4, 3^4)$, then π is one of the following: $(5^3, 4^3, 3)$, $(7^3, 6, 3^5)$, $(5^2, 4, 3^6)$, $(5, 4^3, 3^5)$, $(4^5, 3^4)$, $(5, 4, 3^7)$, $(4^3, 3^6)$, $(5, 4^2, 3^7)$, $(4^4, 3^6)$, $(4^2, 3^8)$, $(7, 5^2, 3^5)$, $(7, 5, 4^2, 3^4)$, $(8, 5^2, 3^6)$, $(8, 5, 4^2, 3^5)$, $(7, 6^2, 5, 3^4)$, $(6, 5, 4, 3^5)$, $(6, 4^3, 3^4)$, $(7, 5, 4, 3^6)$, $(7, 4^3, 3^5)$, $(5, 4^2, 3^5)$, $(4^4, 3^4)$, $(6, 4^2, 3^6)$. It is easy to check that all of these are potentially $K_{3,3}$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_4 \geq 3$ and $d'_{n-1} \geq 2$. If π' satisfies (1)-(3) and (6)-(9), then by the induction hypothesis, π' is potentially $K_{3,3}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_6 = 2$, then $\pi = (d_1, 3^5, 2^{n-6})$ where d_1 is odd. We will show that π is potentially $K_{3,3}$-graphic. If $d_1 = 3$, then $\pi = (3^6, 2^{n-6})$. Since $\pi \neq (3^6, 2)$ and $(3^6, 2^2)$, we have $n \geq 9$. Clearly, $K_{3,3} \cup C_{n-6}$ is a realization of π. In other words, $(3^6, 2^{n-6})$ where $n \geq 9$ is potentially $K_{3,3}$-graphic. If $d_1 \geq 5$, then by π satisfying (2), we have $d_1 \leq n-3$. It is enough to show $\pi_1 = (d_1 - 3, 2^{n-6})$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-3-d_1}, 1^{d_1-3})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

If π' does not satisfy (2), i.e., $d'_1 = n-1-i$ and $d'_{4-i} = 3$ for $i = 1, 2$. If $d'_1 = n-2$ and $d'_3 = 3$, then $d_1 = n-1$, by π satisfying (2), we have $d_2 = d_3 = 4$ and $d_4 = d_5 = d_6 = 3$. Hence, $\pi = (n-1, 4^2, 3^k, 2^{n-3-k})$ where $k \geq 3$, $n-3-k \geq 1$, n and k have different parities. Since $\pi'_1 = (3^2, 2^k, 1^{n-3-k})$ is potentially $K_{2,3}$-graphic by Theorem 2.3, π is potentially $K_{3,3}$-graphic. If $d'_1 = n-3$ and $d'_2 = 3$, then $d_1 = n-2$, $d_2 = 4$ and $d_3 = d_4 = d_5 = d_6 = 3$. Hence, $\pi = (n-2, 4, 3^k, 2^{n-2-k})$ where $k \geq 4$, $n-2-k \geq 1$, n and k have the same parity. We will show that π is potentially $K_{3,3}$-graphic. It is enough to show $\pi_1 = (n-5, 3^{k-4}, 2^{n-2-k}, 1)$ is graphic. It clearly suffices to show $\pi_2 = (2^{k-4}, 1^{n-2-k})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

If π' does not satisfy (3), i.e., $d'_2 = n-2$, $d'_3 = 4$ and $d'_6 = 3$. If $n \geq 8$, then $d_2 = n-1$, $d_3 = 4$ and $d_6 = 3$, which contradicts condition (3). If $n = 7$, then $\pi' = (5^2, 4^2, 3^2)$. Since $\pi \neq (6^2, 4^2, 3^2, 2)$ and $(5^4, 3^2, 2)$, then $\pi = (6, 5^2, 4, 3^2, 2)$, which is potentially $K_{3,3}$-graphic.

If π' does not satisfy (6), then $\pi' = (d'_1, d'_2, 3^4, 2^{n-7})$ or $(d'_1, d'_2, 4^2, 3^2, 2^{n-7})$, and $d'_1 + d'_2 > 2n-6$. If $\pi' = (d'_1, d'_2, 3^4, 2^{n-7})$, then $d_1 + d_2 = d'_1 + d'_2 + 2 > 2n-4$, a contradiction. If $\pi' = (d'_1, d'_2, 4^2, 3^2, 2^{n-7})$ and $n \geq 8$, then

$d_1+d_2 = d'_1+d'_2+2 > 2n-4$, a contradiction. If $n = 7$, then $\pi' = (5^2, 4^2, 3^2)$. Since $\pi \neq (6^2, 4^2, 3^2, 2)$ and $(5^4, 3^2, 2)$, we have $\pi = (6, 5^2, 4, 3^2, 2)$. It is easy to check that π is potentially $K_{3,3}$-graphic.

If π' does not satisfy (7), then $\pi' = (d'_1, d'_2, 4, 3^4, 2^{n-8})$ and $d'_1 + d'_2 > 2n - 6$. Hence, $d_1 + d_2 \geq d'_1 + d'_2 + 2 > 2n - 4$, a contradiction.

If π' does not satisfy (8), then $\pi' = ((n-2)^2, 4^5, 2^{n-8})$ or $((n-2)^2, 4^6, 2^{n-9})$. Hence, $\pi = ((n-1)^2, 4^5, 2^{n-7})$ or $((n-1)^2, 4^6, 2^{n-8})$, a contradiction.

If π' does not satisfy (9), then π' is one of the following:$(5^4, 3^2, 2)$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(4, 3^8)$, $(3^8, 2)$, $(6, 4^2, 3^4)$, $(7, 4^2, 3^5)$, $(6, 5^3, 3^3)$, $(5, 4, 3^5)$, $(6, 4, 3^6)$, $(4, 3^6)$, $(5, 3^7)$. Since $\pi \neq (4, 3^6, 2)$ and $(3^8, 2)$, then π is one of the following:$(6^2, 5^2, 3^2, 2)$, $(5^2, 4^4, 2)$, $(4^2, 3^4, 2^2)$, $(7^2, 6^2, 3^4, 2)$, $(5^2, 3^6, 2)$, $(5, 4^2, 3^5, 2)$, $(4^4, 3^4, 2)$, $(5, 4, 3^5, 2^2)$, $(4^3, 3^4, 2^2)$, $(4^2, 3^4, 2^3)$, $(4, 3^6, 2^2)$,$(4^2, 3^6, 2)$,$(5, 4, 3^7, 2)$,$(4^3, 3^6, 2)$, $(4^2, 3^6, 2^2)$,$(4, 3^8, 2)$, $(7, 5, 4, 3^4, 2)$, $(7, 4^3, 3^3, 2)$, $(8, 5, 4, 3^5, 2)$, $(8, 4^3, 3^4, 2)$, $(7, 6, 5^2, 3^3, 2)$, $(6^3, 5, 3^3, 2)$, $(6, 5, 3^5, 2)$, $(6, 4^2, 3^4, 2)$, $(7, 5, 3^6, 2)$, $(7, 4^2, 3^5, 2)$, $(5, 4, 3^5, 2)$, $(4^3, 3^4, 2)$, $(6, 4, 3^6, 2)$. It is easy to check that all of these are potentially $K_{3,3}$-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_5 \geq 3$ and $d'_6 \geq 2$. If π' satisfies (1)-(9), then by the induction hypothesis, π' is potentially $K_{3,3}$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_6 = 2$, then $\pi = (3^6, 2^k, 1^{n-6-k})$ where $n - 6 - k \geq 1$ and $n - 6 - k$ is even. We will show that π is potentially $K_{3,3}$-graphic. It is enough to show $\pi_1 = (2^k, 1^{n-6-k})$ is graphic. By $\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

If π' does not satisfy (2), i.e., $d'_1 = n - 1 - i$ and $d'_{4-i} = 3$ for $i = 1, 2$. If $d'_1 = n - 2$ and $d'_3 = 3$, then $d_1 = n - 1$, $d_3 = 3$ or $d_1 = d_2 = n - 2$, $d_3 = 3$, which contradicts condition (2) and (5), respectively. If $d'_1 = n - 3$ and $d'_2 = 3$, then $d_1 = n - 2$ and $d_2 = 3$, which is also a contradiction.

If π' does not satisfy (3), i.e., $d'_2 = n - 2$, $d'_3 \leq 4$ and $d'_6 = 3$. If $n \geq 8$, then $d_1 = n - 1$, $d_2 = n - 2$, $3 \leq d_3 \leq 4$ and $d_6 = 3$, which contradicts condition (4). If $n = 7$, then $\pi' = (5^2, 3^4)$ or $(5^2, 4^2, 3^2)$. By π satisfying (2) and (4), we have $\pi = (5^3, 4, 3^2, 1)$, which is potentially $K_{3,3}$-graphic.

If π' does not satisfy (4), i.e., $d'_1 + d'_2 = 2n - 2 - i$, $d'_{n-i+2} = 1$, $d'_3 \leq 4$ and $d'_6 = 3$. Then $d_1 + d_2 = 2n - (i + 1)$, $d_{n-(i+1)+3} = 1$, $3 \leq d_3 \leq 4$ and $d_6 = 3$, which is a contradiction. Similarly, one can check that π' also satisfies (5).

If π' does not satisfy (6), i.e., $\pi' = (d'_1, d'_2, 3^4, 2^t, 1^{n-7-t})$ or

$(d_1', d_2', 4^2, 3^2, 2^t, 1^{n-7-t})$, and $d_1' + d_2' > n+t+1$. Then $d_1 + d_2 > n+t+2$, a contradiction. Similarly, one can show that π' satisfies (7).

If π' does not satisfy (8), i.e., $\pi' = (n-1-i, k+i, 4^t, 2^{k-t}, 1^{n-3-k})$ for $t = 5, 6$. If $\pi' = (n-1-i, k+i, 4^5, 2^{k-5}, 1^{n-3-k})$ and $n-1-i > k+i+1$ or $n-1-i = k+i$, then $\pi = (n-i, k+i, 4^5, 2^{k-5}, 1^{n-2-k})$, a contradiction. If $n-1-i = k+i+1$, i.e., $\pi' = (n-1-i, n-2-i, 4^5, 2^{n-7-2i}, 1^{2i-1})$, then $\pi = (n-i, n-2-i, 4^5, 2^{n-7-2i}, 1^{2i})$ or $((n-1-i)^2, 4^5, 2^{n-7-2i}, 1^{2i})$, which also contradicts condition (8). Similarly, one can show that $\pi \neq (n-i, k+i, 4^6, 2^{k-6}, 1^{n-2-k})$.

If π' does not satisfy (9), since $\pi \neq (4, 3^7, 1)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-2, 4, 3^6, 1^{n-8})$ and $(n-3, 3^7, 1^{n-8})$, then π' is one of the following:$(5^4, 3^2, 2)$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(6, 5^3, 3^3)$, $(5, 4, 3^5)$, $(4, 3^6)$. By $\pi \neq (3^7, 1)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-3, 3^6, 1^{n-7})$, π is one of the following:$(6, 5^3, 3^2, 2, 1)$, $(5, 4^5, 1)$, $(4, 3^5, 2, 1)$, $(7, 6^3, 3^4, 1)$, $(5, 4, 3^6, 1)$, $(4^3, 3^5, 1)$, $(5, 3^6, 2, 1)$, $(4^2, 3^5, 2, 1)$, $(4, 3^5, 2^2, 1)$, $(4, 3^6, 1^2)$, $(5, 3^8, 1)$, $(4^2, 3^7, 1)$,$(5, 3^7, 1^2)$,$(4^2, 3^6, 1^2)$, $(4, 3^7, 2, 1)$,$(4, 3^6, 2, 1^2)$, $(4, 3^8, 1^2)$, $(4, 3^7, 1^3)$, $(6^2, 5^2, 3^3, 1)$, $(5^2, 3^5, 1)$, $(4^2, 3^5, 1)$. It is easy to check that all of these are potentially $K_{3,3}$-graphic.

Theorem 3.2 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 6$. Then π is potentially $K_6 - C_6$-graphic if and only if the following conditions hold:

(1) $d_6 \geq 3$;

(2) For $i = 1, 2$, $d_1 = n - i$ implies $d_{4-i} \geq 4$;

(3) $d_2 = n - 1$ implies $d_4 \geq 4$;

(4) $d_1 + d_2 = 2n - i$ and $d_{n-i+3} = 1 (3 \leq i \leq n-4)$ implies $d_4 \geq 4$;

(5) $d_1 + d_2 = 2n - i$ and $d_{n-i+4} = 1 (4 \leq i \leq n-3)$ implies $d_3 \geq 4$;

(6) $\pi = (d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$ implies $d_1 + d_2 + d_3 \leq n + 2k + t + 1$;

(7) $\pi = (d_1, d_2, 3^4, 2^t, 1^{n-6-t})$ implies $d_1 + d_2 \leq n + t + 2$;

(8) $\pi \neq (n-i, k, t, 3^t, 2^{k-i-t-1}, 1^{n-2-k+i})$ where $i = 1, \cdots, [\frac{n-t-1}{2}]$ and $k = i + t + 1, \cdots, n - i$ and $t = 4, 5, \cdots, k - i - 1$;

(9) $\pi \neq (3^6, 2)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$, $(n-3, 3^7, 1^{n-8})$.

Proof: First we show the conditions (1)-(9) are necessary conditions for π to be potentially $K_6 - C_6$-graphic. Assume that π is potentially

$K_6 - C_6$-graphic. With the same argument as $K_{3,3}$, one can check that π satisfies conditions (1)-(5) and (7),(9). Now we show that π also satisfies (6) and (8). If $\pi = (d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$ is potentially $K_6 - C_6$-graphic, then according to Theorem 2.1, there exists a realization G of π containing K_6-C_6 as a subgraph so that the vertices of K_6-C_6 have the largest degrees of π. Therefore, the sequence $\pi_1 = (d_1-3, d_2-3, d_3-3, 3^{k-3}, 2^t, 1^{n-3-k-t})$ obtained from $G-(K_6-C_6)$ must be graphic. It follows $d_1 - 3 + d_2 - 3 + d_3 - 3 - 4 \leq 3(k-3) + 2t + n - 3 - k - t$, i.e., $d_1 + d_2 + d_3 \leq n + 2k + t + 1$. Hence, (6) holds. If $\pi = (n-i, k, t, 3^t, 2^{k-i-t-1}, 1^{n-2-k+i})$ is potentially $K_6 - C_6$-graphic, then according to Theorem 2.1, there exists a realization G of π containing K_6-C_6 as a subgraph so that the vertices of K_6-C_6 have the largest degrees of π. Therefore, the sequence $\pi_2 = (n-i-3, k-3, t-3, 3^{t-3}, 2^{k-i-t-1}, 1^{n-2-k+i})$ obtained from $G-(K_6-C_6)$ must be graphic. It follows $n-i-3+k-3+t-3+3(t-3)-2(3t-8) \leq 2k-2i-2t-2+n-2-k+i$, i.e., $-2 \leq -4$, a contradiction. Hence, (8) holds.

Now we prove the sufficient conditions. Suppose the graphic sequence π satisfies the conditions (1)-(9). Our proof is by induction on n. We first prove the base case where $n = 6$. In this case, π is one of the following: (5^6), $(5^4, 4^2)$, $(5^3, 4^2, 3)$, $(5^2, 4^4)$, $(5^2, 4^2, 3^2)$, $(5, 4^4, 3)$, $(5, 4^2, 3^3)$, (4^6), $(4^4, 3^2)$, $(4^2, 3^4)$, (3^6). It is easy to check that all of these are potentially $K_6 - C_6$-graphic. Now suppose that the sufficiency holds for $n-1 (n \geq 7)$, we will show that π is potentially $K_6 - C_6$-graphic in terms of the following cases:

Case 1: $d_n \geq 4$. It is easy to check that $\pi' = (d'_1, d'_2, \cdots, d'_n)$ satisfies (1)-(9), then by the induction hypothesis, π' is potentially K_6-C_6-graphic, and hence so is π.

Case 2: $d_n = 3$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_{n-4} \geq 3$ and $d'_{n-1} \geq 2$. With the same argument as $K_{3,3}$, one can check that π' satisfies (1) and (7). If π' also satisfies (2), (3), (6) and (8)-(9), then by the induction hypothesis, π' is potentially $K_6 - C_6$-graphic, and hence so is π.

If π' does not satisfy (2), i.e., $d'_1 = n-1-i$ and $d'_{4-i} = 3$ for $i = 1, 2$. If $d'_1 = n-2$ and $d'_3 = 3$, then $d_1 = n-1$ and $d_3 = 4$. Since $\sigma(\pi)$ is even, we have $d_4 = 3$. Hence, $\pi = (n-1, d_2, 4, 3^{n-3})$ where $4 \leq d_2 \leq n-2$, n and d_2 have different parities. If $d_2 = 4$, then $\pi = (n-1, 4^2, 3^{n-3})$. By $\pi \neq (6, 4^2, 3^4)$ and $(7, 4^2, 3^5)$, we have $n \geq 9$. Since $\pi'_1 = (3^2, 2^{n-3})$ is potentially $K_5 - P_4$-graphic by Theorem 2.4, π is potentially $K_6 - C_6$-graphic. If $5 \leq d_2 \leq n-2$, then $\pi'_1 = (d_2 - 1, 3, 2^{n-3})$ is also potentially $K_5 - P_4$-graphic by Theorem 2.4. Hence, π is potentially $K_6 - C_6$-graphic.

If $d'_1 = n-3$ and $d'_2 = 3$, then $d_1 = n-2$, $d_2 = 4$ and $3 \le d_3 \le 4$. Since $\sigma(\pi)$ is even, $d_3 = 3$. Hence, $\pi = (n-2, 4, 3^{n-2})$ where n is arbitrary. Since $\pi \ne (5,4,3^5)$ and $(6,4,3^6)$, we have $n \ge 9$. We will show that π is potentially $K_6 - C_6$-graphic. It is enough to show $\pi_1 = (n-5, 3^{n-6}, 1)$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-6})$ is graphic. Clearly, C_{n-6} is a realization of π_2.

If π' does not satisfy (3), i.e., $d'_2 = n-2$ and $d'_4 = 3$. It is easy to check that $d_1 = d_2 = n-1$ and $3 \le d_4 \le 4$. By π satisfying (3), we have $d_4 = 4$. Hence, $\pi = ((n-1)^2, 4^2, 3^{n-4})$ where n is even. Since $\pi'_1 = (n-2, 3^2, 2^{n-4})$ is potentially $K_5 - P_4$-graphic by Theorem 2.4, π is potentially $K_6 - C_6$-graphic.

If π' does not satisfy (6), then $\pi' = (d'_1, d'_2, d'_3, 3^{n-4})$ and $d'_1 + d'_2 + d'_3 > n-1+2(n-4)+1 = 3n-8$. Hence, $d_1 + d_2 + d_3 = d'_1 + d'_2 + d'_3 + 3 > 3n-5$, a contradiction.

If π' does not satisfy (8), then $\pi' = ((n-2)^2, n-4, 3^{n-4})$. If $n = 7$ or $n \ge 9$, then $\pi = ((n-1)^2, n-3, 3^{n-3})$, a contradiction. If $n = 8$, i.e., $\pi' = (6^2, 4, 3^4)$, then $\pi = (7^2, 5, 3^5)$ or $(7^2, 4^2, 3^4)$. By π satisfying (3), we have $\pi = (7^2, 4^2, 3^4)$, which is potentially $K_6 - C_6$-graphic.

If π' does not satisfy (9), since $\pi \ne (4^2, 3^6)$ and $(4, 3^8)$, then π' is one of the following: $(4^2, 3^6)$, $(4, 3^6, 2)$, (3^8), $(4, 3^8)$, $(3^8, 2)$, $(6, 4^2, 3^4)$, $(7, 4^2, 3^5)$, $(5, 4, 3^5)$, $(6, 4, 3^6)$, $(4, 3^6)$, $(5, 3^7)$. Hence, π is one of the following: $(5^2, 4, 3^6)$, $(5, 4^3, 3^5)$, $(4^5, 3^4)$, $(5, 4, 3^7)$, $(4^3, 3^6)$, $(5, 4^2, 3^7)$, $(4^4, 3^6)$, $(4^2, 3^8)$, $(7, 5^2, 3^5)$, $(7, 5, 4^2, 3^4)$, $(8, 5^2, 3^6)$, $(8, 5, 4^2, 3^5)$, $(6, 5, 4, 3^5)$, $(6, 4^3, 3^4)$, $(7, 5, 4, 3^6)$, $(7, 4^3, 3^5)$, $(5, 4^2, 3^5)$, $(4^4, 3^4)$, $(6, 4^2, 3^6)$. It is easy to check that all of these are potentially $K_6 - C_6$-graphic.

Case 3: $d_n = 2$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_4 \ge 3$ and $d'_{n-1} \ge 2$. With the same argument as $K_{3,3}$, one can check that π' satisfies (7). If π' also satisfies (1)-(3), (6) and (8)-(9), then by the induction hypothesis, π' is potentially $K_6 - C_6$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_6 = 2$, then $\pi = (d_1, 3^5, 2^{n-6})$ where d_1 is odd. We will show that π is potentially $K_6 - C_6$-graphic. If $d_1 = 3$, then $\pi = (3^6, 2^{n-6})$. Since $\pi \ne (3^6, 2)$ and $(3^6, 2^2)$, we have $n \ge 9$. Clearly, $K_6 - C_6 \cup C_{n-6}$ is a realization of π. In other words, $(3^6, 2^{n-6})$ where $n \ge 9$ is potentially $K_6 - C_6$-graphic. If $d_1 \ge 5$, then by π satisfying (2), we have $d_1 \le n-3$. It is enough to show $\pi_1 = (d_1 - 3, 2^{n-6})$ is graphic. It clearly suffices to show $\pi_2 = (2^{n-3-d_1}, 1^{d_1-3})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

If π' does not satisfy (2), i.e., $d'_1 = n-1-i$ and $d'_{4-i} = 3$ for $i = 1, 2$. If $d'_1 = n-2$ and $d'_3 = 3$, then $d_1 = n-1$. By π satisfying (2), we have $d_2 = d_3 = 4$ and $d_4 = d_5 = d_6 = 3$. Hence, $\pi = (n-1, 4^2, 3^k, 2^{n-3-k})$ where $k \geq 3$, $n-3-k \geq 1$, n and k have different parities. Since $\pi'_1 = (3^2, 2^k, 1^{n-3-k})$ is potentially $K_5 - P_4$-graphic by Theorem 2.4, π is potentially $K_6 - C_6$-graphic. If $d'_1 = n-3$ and $d'_2 = 3$, then $d_1 = n-2$, $d_2 = 4$ and $d_3 = d_4 = d_5 = d_6 = 3$. Hence, $\pi = (n-2, 4, 3^k, 2^{n-2-k})$ where $k \geq 4$, $n-2-k \geq 1$, n and k have the same parity. We will show that π is potentially $K_6 - C_6$-graphic. It is enough to show $\pi_1 = (n-5, 3^{k-4}, 2^{n-2-k}, 1)$ is graphic. It clearly suffices to show $\pi_2 = (2^{k-4}, 1^{n-2-k})$ is graphic. By $\sigma(\pi_2)$ being even and Theorem 2.2, π_2 is graphic.

If π' does not satisfy (3), i.e., $d'_2 = n-2$ and $d'_4 = 3$. Then $d_1 = n-1$, $d_2 = n-1$ or $n-2$ and $d_4 = 3$. By π satisfying (3), we have $d_2 = n-2$. Hence, $\pi = (n-1, (n-2)^2, 3^k, 2^{n-3-k})$ where $k \geq 3$, $n-3-k \geq 1$, and, n and k have different parities. By π satisfying (6), we have $n-1+2(n-2) \leq n+2k+n-3-k+1$, i.e., $n \leq k+3$, a contradiction.

If π' does not satisfy (6), then $\pi' = (d'_1, d'_2, d'_3, 3^k, 2^{n-4-k})$ and $d'_1 + d'_2 + d'_3 > n-1+2k+n-4-k+1 = 2n+k-4$. Hence, $d_1 + d_2 + d_3 = d'_1 + d'_2 + d'_3 + 2 > 2n+k-2$, a contradiction.

If π' does not satisfy (8), then $\pi' = ((n-2)^2, t, 3^t, 2^{n-4-t})$ where $t = 4, \cdots, n-4$. Hence, $\pi = ((n-1)^2, t, 3^t, 2^{n-3-t})$, a contradiction.

If π' does not satisfy (9), then π' is one of the following: $(3^6, 2)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(4, 3^8)$, $(3^8, 2)$, $(6, 4^2, 3^4)$, $(7, 4^2, 3^5)$, $(5, 4, 3^5)$, $(6, 4, 3^6)$, $(4, 3^6)$, $(5, 3^7)$. Since $\pi \neq (4, 3^6, 2)$ and $(3^8, 2)$, then π is one of the following: $(4^2, 3^4, 2^2)$, $(5^2, 3^6, 2)$, $(5, 4^2, 3^5, 2)$, $(4^4, 3^4, 2)$, $(5, 4, 3^5, 2^2)$, $(4^3, 3^4, 2^2)$, $(4^2, 3^4, 2^3)$, $(4, 3^6, 2^2)$, $(4^2, 3^6, 2)$, $(5, 4, 3^7, 2)$, $(4^3, 3^6, 2)$, $(4^2, 3^6, 2^2)$, $(4, 3^8, 2)$, $(7, 5, 4, 3^4, 2)$, $(7, 4^3, 3^3, 2)$, $(8, 5, 4, 3^5, 2)$, $(8, 4^3, 3^4, 2)$, $(6, 5, 3^5, 2)$, $(6, 4^2, 3^4, 2)$, $(7, 5, 3^6, 2)$, $(7, 4^2, 3^5, 2)$, $(5, 4, 3^5, 2)$, $(4^3, 3^4, 2)$, $(6, 4, 3^6, 2)$. It is easy to check that all of these are potentially $K_6 - C_6$-graphic.

Case 4: $d_n = 1$. Consider $\pi' = (d'_1, d'_2, \cdots, d'_{n-1})$ where $d'_5 \geq 3$ and $d'_6 \geq 2$. With the same argument as $K_{3,3}$, one can check that π' satisfies (2) and (4)-(8). If π' also satisfies other conditions in Theorem 3.2, then by the induction hypothesis, π' is potentially $K_6 - C_6$-graphic, and hence so is π.

If π' does not satisfy (1), i.e., $d'_6 = 2$, then $\pi = (3^6, 2^k, 1^{n-6-k})$ where $n-6-k \geq 1$ and $n-6-k$ is even. We will show that π is potentially $K_6 - C_6$-graphic. It is enough to show $\pi_1 = (2^k, 1^{n-6-k})$ is graphic. By

$\sigma(\pi_1)$ being even and Theorem 2.2, π_1 is graphic.

If π' does not satisfy (3), i.e., $d'_2 = n-2$ and $d'_4 = 3$. There are two subcases.

Subcase1: $d_1 = n-1$, $d_2 = n-2$ and $d_4 = 3$, which contradicts condition (4).

Subcase2: $d_1 = d_2 = d_3 = n-2$ and $d_4 = 3$. Then $\pi = ((n-2)^3, 3^k, 2^t, 1^{n-3-k-t})$ where $k \geq 3$, $n-3-k-t \geq 1$ and t is odd. If $n-3-k-t \geq 2$, then π contradicts condition (4). Hence, we may assume $\pi = ((n-2)^3, 3^k, 2^{n-4-k}, 1)$. By π satisfying (6), we have $3(n-2) \leq n+2k+n-4-k+1$, i.e., $n \leq k+3$, a contradiction.

If π' does not satisfy (9), since $\pi \neq (4, 3^7, 1)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-2, 4, 3^6, 1^{n-8})$, and $(n-3, 3^7, 1^{n-8})$, then π' is one of the following: $(3^6, 2)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(5, 4, 3^5)$, $(4, 3^6)$. By $\pi \neq (3^7, 1)$, $(3^7, 2, 1)$, $(3^9, 1)$, $(3^8, 1^2)$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-3, 3^6, 1^{n-7})$, π is one of the following: $(4, 3^5, 2, 1)$, $(5, 4, 3^6, 1)$, $(4^3, 3^5, 1)$, $(5, 3^6, 2, 1)$, $(4^2, 3^5, 2, 1)$, $(4, 3^5, 2^2, 1)$, $(4, 3^6, 1^2)$, $(5, 3^8, 1)$, $(4^2, 3^7, 1)$, $(5, 3^7, 1^2)$, $(4^2, 3^6, 1^2)$, $(4, 3^7, 2, 1)$, $(4, 3^6, 2, 1^2)$, $(4, 3^8, 1^2)$, $(4, 3^7, 1^3)$, $(5^2, 3^5, 1)$, $(4^2, 3^5, 1)$. It is easy to check that all of these are potentially $K_6 - C_6$-graphic.

4 Application

In the remaining of this section, we will use the above two theorems to find exact values of $\sigma(K_{3,3}, n)$ and $\sigma(K_6 - C_6, n)$. Note that the value of $\sigma(K_{3,3}, n)$ was determined by Yin in [25] so a much simpler proof is given here.

Theorem 4.1 (Yin [25]) If $n \geq 11$, then

$$\sigma(K_{3,3}, n) = \begin{cases} 5n-3, & \text{if } n \text{ is odd,} \\ 5n-4, & \text{if } n \text{ is even.} \end{cases}$$

Proof: First we claim that for $n \geq 11$,

$$\sigma(K_{3,3}, n) \geq \begin{cases} 5n-3, & \text{if } n \text{ is odd,} \\ 5n-4, & \text{if } n \text{ is even.} \end{cases}$$

If n is odd, take $\pi_1 = ((n-1)^2, 4^3, 3^{n-5})$, then $\sigma(\pi_1) = 5n-5$, and it is easy to see that π_1 is not potentially $K_{3,3}$-graphic by Theorem 3.1. If n is

even, take $\pi_1 = ((n-1)^2, 4^3, 3^{n-6}, 2)$, then $\sigma(\pi_1) = 5n-6$, and it is easy to see that π_1 is not potentially $K_{3,3}$-graphic by Theorem 3.1. Thus,

$$\sigma(K_{3,3}, n) \geq \begin{cases} \sigma(\pi_1) + 2 = 5n-3, & \text{if } n \text{ is odd,} \\ \sigma(\pi_1) + 2 = 5n-4, & \text{if } n \text{ is even.} \end{cases}$$

Now we show that if π is an n-term ($n \geq 11$) graphical sequence with $\sigma(\pi) \geq 5n-4$, then there exists a realization of π containing $K_{3,3}$. Hence, it suffices to show that π is potentially $K_{3,3}$-graphic.

If $d_6 \leq 2$, then $\sigma(\pi) \leq d_1 + d_2 + d_3 + d_4 + d_5 + 2(n-5) \leq 20 + 2(n-5) + 2(n-5) = 4n < 5n-4$, a contradiction. Hence, $d_6 \geq 3$.

If $d_1 = n-1$ and $d_3 \leq 3$, then $\sigma(\pi) \leq d_1 + d_2 + 3(n-2) \leq 2(n-1) + 3(n-2) = 5n-8 < 5n-4$, a contradiction. If $d_1 = n-2$ and $d_2 \leq 3$, then $\sigma(\pi) \leq d_1 + 3(n-1) \leq (n-2) + 3(n-1) = 4n-5 < 5n-4$, a contradiction. Hence, $d_1 = n-i$ implies $d_{4-i} \geq 4$ for $i = 1, 2$.

If $d_2 = n-1$ and $d_3 = 4$, $d_6 = 3$, then $\sigma(\pi) \leq 2(n-1) + 3 \times 4 + 3(n-5) = 5n-5 < 5n-4$, a contradiction. Hence, $d_2 = n-1$ implies $d_3 \geq 5$ or $d_6 \geq 4$.

If $d_1 + d_2 = 2n-i$, $d_{n-i+3} = 1 (3 \leq i \leq n-4)$ and $d_3 \leq 4$, $d_6 = 3$, then $\sigma(\pi) \leq 2n - i + 4 \times 3 + 3(n-3-i) + i - 2 = 5n - (3i-1) < 5n-4$, a contradiction. Hence, $d_1 + d_2 = 2n-i$ and $d_{n-i+3} = 1$ implies $d_3 \geq 5$ or $d_6 \geq 4$.

If $d_1 + d_2 = 2n-i$, $d_{n-i+4} = 1 (4 \leq i \leq n-3)$ and $d_3 = 3$, then $\sigma(\pi) \leq 2n - i + 3(n+1-i) + i - 3 = 5n - 3i < 5n-4$, a contradiction. Hence, $d_1 + d_2 = 2n-i$ and $d_{n-i+4} = 1$ implies $d_3 \geq 4$.

Since $\sigma(\pi) \geq 5n-4$, then π is not one of the following: $(d_1, d_2, 3^4, 2^t, 1^{n-6-t})$, $(d_1, d_2, 4^2, 3^2, 2^t, 1^{n-6-t})$, $(d_1, d_2, 4, 3^4, 2^t, 1^{n-7-t})$ $(n-i, k+i, 4^t, 2^{k-t}, 1^{n-2-k})$ where $t = 5, 6$, (4^6), $(3^6, 2)$, $(6^4, 3^4)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$,$(3^8, 1^2)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-1, 5^3, 3^3, 1^{n-7})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$, $(n-3, 3^7, 1^{n-8})$. Thus, π satisfies the conditions (1)-(9) in Theorem 3.1. Therefore, π is potentially $K_{3,3}$-graphic.

Corollary 4.2 For $n \geq 6$, $\sigma(K_6 - C_6, n) = 6n-10$.

Proof. First we claim $\sigma(K_6 - C_6, n) \geq 6n-10$ for $n \geq 6$. We would like to show there exists π_1 with $\sigma(\pi_1) = 6n-12$ such that π_1 is not potentially $K_6 - C_6$-graphic. Let $\pi_1 = ((n-1)^3, 3^{n-3})$. It is easy to see that $\sigma(\pi_1) = 6n-12$ and π_1 is not potentially $K_6 - C_6$-graphic by Theorem 3.2.

Now we show if π is an n-term ($n \geq 6$) graphic sequence with $\sigma(\pi) \geq 6n-10$, then there exists a realization of π containing a K_6-C_6. If $d_6 \leq 2$, then $\sigma(\pi) \leq d_1+d_2+d_3+d_4+d_5+2(n-5) \leq 20+2(n-5)+2(n-5) = 4n < 6n-10$, a contradiction. Hence, $d_6 \geq 3$.

If $d_1 = n-1$ and $d_3 \leq 3$, then $\sigma(\pi) \leq d_1+d_2+3(n-2) \leq 2(n-1)+3(n-2) = 5n-8 < 6n-10$, a contradiction. If $d_1 = n-2$ and $d_2 \leq 3$, then $\sigma(\pi) \leq d_1+3(n-1) \leq (n-2)+3(n-1) = 4n-5 < 6n-10$, a contradiction. Hence, $d_1 = n-i$ implies $d_{4-i} \geq 4$ for $i = 1, 2$.

If $d_2 = n-1$ and $d_4 \leq 3$, then $\sigma(\pi) \leq 3(n-1)+3(n-3) = 6n-12 < 6n-10$, a contradiction. Hence, $d_2 = n-1$ implies $d_4 \geq 4$.

If $d_1+d_2 = 2n-i$, $d_{n-i+3} = 1(3 \leq i \leq n-4)$ and $d_4 = 3$, then $\sigma(\pi) \leq 2n-i+n-2+3(n-1-i)+i-2 = 6n-(3i+7) < 6n-10$, a contradiction. Hence, $d_1+d_2 = 2n-i$ and $d_{n-i+3} = 1$ implies $d_4 \geq 4$.

If $d_1+d_2 = 2n-i$, $d_{n-i+4} = 1(4 \leq i \leq n-3)$ and $d_3 = 3$, then $\sigma(\pi) \leq 2n-i+3(n+1-i)+i-3 = 5n-3i < 6n-10$, a contradiction. Hence, $d_1+d_2 = 2n-i$ and $d_{n-i+4} = 1$ implies $d_3 \geq 4$.

Since $\sigma(\pi) \geq 6n-10$, then π is not one of the following: $(d_1, d_2, d_3, 3^k, 2^t, 1^{n-3-k-t})$, $(d_1, d_2, 3^4, 2^t, 1^{n-6-t})$, $(n-i, k, t, 3^t, 2^{k-i-t-1}, 1^{n-2-k+i})$, $(3^6, 2)$, $(4^2, 3^6)$, $(4, 3^6, 2)$, $(3^6, 2^2)$, (3^8), $(3^7, 1)$, $(4, 3^8)$, $(4, 3^7, 1)$, $(3^8, 2)$, $(3^7, 2, 1)$, $(3^9, 1)$,$(3^8, 1^2)$, $(n-1, 4^2, 3^4, 1^{n-7})$, $(n-1, 4^2, 3^5, 1^{n-8})$, $(n-2, 4, 3^5, 1^{n-7})$, $(n-2, 4, 3^6, 1^{n-8})$, $(n-3, 3^6, 1^{n-7})$, $(n-3, 3^7, 1^{n-8})$. Thus, π satisfies the conditions (1)-(9) in Theorem 3.2. Therefore, π is potentially K_6-C_6-graphic.

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